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FREE TORSIONAL VIBRATIONS OF A HOLLOW CYLINDER WITH LAMINATED PERIODIC STRUCTURE†

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Abstract—The theory of torsional vibrations of a circular, hollow cylinder with a piecewise constant periodic variation of rigidity modulus and mass density is developed in terms of Floquet waves. The dispersion spectrum is shown to have a band structure, and the arrangement of characteristic sequence at the end-points of the Brillouin zone is studied. The problem of *co-existence* of periodic solution is examined in detail and the regions of stability and lability are charted.

1. INTRODUCTION AND GOVERNING EQUATIONS

Consider an infinite, circular, hollow cylinder with inner radius b, outer radius a and thickness $2h \equiv (a-b)$, $a > b$, $b \geq 0$. In cylindrical coordinates (r, θ, z), the domain R of the hollow, homogeneous cylinder is defined as $R: R_1 \times R_2$, where $R_1: b < r < a$, $R_2: |z| < \infty$ and $0 \leq \theta < 2\pi$. In the longitudinal z-direction the domain R_2 is the union of denumerable infinite number of *cells* R_d with identical properties, so that R_2: $\overset{\infty}{\cup} R_d$. Each cell R_d consists of the union of two layers R_l and $R_{\bar{l}}$ with z-domains R_l: $0 < z < l$ and $R_{\bar{l}}$: $l < z < d$, where $d \equiv (l + \bar{l})$ is the lattice distance of a unit cell. We assume that the material properties of the elastic layer R_l are the modulus of rigidity μ and mass density ρ, and the corresponding material properties of the elastic layer $R_{\bar{l}}$ are specified by the parameters $\bar{\mu}$ and $\bar{\rho}$, respectively. We thus assume that each cell consists of two layers of lengths l and $\bar{l}$, and material properties (μ, ρ) and $(\bar{\mu}, \bar{\rho})$, with the additional assumptions $\mu/\rho > 0$, $\bar{\mu}/\bar{\rho} > 0$ and $\mu/\bar{\mu} > 0$. We also assume that an infinite number of these primitive cells are bonded together at their common interfaces $|z| = nd$, $n = 0, 1, 2, \ldots$ so that the domain R defines an infinite, circular, hollow, layered cylinder with periodic structure of material constants, with real period d.

For time-harmonic torsional waves with angular frequency ω, the displacement components $(u_r, u_z) \equiv 0$, and the non-trivial tangential component of displacement $u_\theta \equiv u$ satisfies in an open domain R the boundary eigenvalue problem

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) - \frac{u}{r^2} + \frac{\partial^2 u}{\partial z^2} + \left(\frac{\pi\Omega}{2h}\right)^2 u = 0, \quad \forall_{P \in R} \tag{1}$$

$$\tau_{r\theta}|_{\partial R} = 0, \quad \partial R\text{: } (r = a \text{ and } r = b), \quad |z| < \infty$$

where for a homogeneous, hollow cylinder R: $b < r < a$, $|z| < \infty$, $\Omega \equiv \omega/\omega_s$ is the non-dimensional frequency, $\omega_s \equiv (\pi/2h)c$ is the lowest thickness-shear frequency of an isotropic, homogeneous, infinite plate of half-thickness h, and $c \equiv (\mu/\rho)^{1/2}$ is the speed of the shear wave in an infinite, homogeneous, isotropic elastic medium with shear modulus μ and mass density ρ [1].

Using Bernoulli's method of separation of variables, we get the separated equations

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}R}{\mathrm{d}r}\right) + \frac{1}{r}\left[\left(\frac{\pi\kappa}{2h}r\right)^2 - 1\right]R = 0, \quad b < r < a$$

$$Z'' + \left[\frac{\pi}{2h}\xi\right]^2 Z = 0, \quad |z| < \infty \tag{2}$$

$$\kappa^2 \equiv (\Omega^2 - \xi^2),$$

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where we assumed the product solution to be of the form $u(r, z) \equiv R(r)Z(z)$, and accents indicate differentiation with respect to coordinate z. Since

$$\tau_{r\theta} = \mu r \frac{\partial}{\partial r}(u/r), \tag{3}$$

it follows that the boundary conditions are

$$\frac{d}{dr}(rR) - 2R = 0 \text{ on } \partial R. \tag{4}$$

In addition $\tau_{rr} = \tau_{rz} \equiv 0$ and

$$\tau_{z\theta} = \mu \frac{\partial u}{\partial z}. \tag{5}$$

Now the structured, hollow cylinder consists of an infinite number of primitive cells, and in the case under discussion, each cell consists of two layers. Therefore, for the two layers in each cell, the governing equations are

$$Z'' + \left(\frac{\pi}{2h}\xi\right)^2 Z = 0, \ \xi^2 = (\Omega^2 - \kappa^2), \ 0 < z < l$$

$$\bar{Z}'' + \left(\frac{\pi}{2h}\bar{\xi}\right)^2 \bar{Z} = 0, \ \bar{\xi}^2 = ((\gamma\Omega)^2 - \kappa^2), \ l < z < d$$

where ξ and $\bar{\xi}$ are the longitudinal wave numbers in the layers of thickness l and $\bar{l}$, respectively, and $\gamma^2 \equiv (\mu/\rho)(\bar{\rho}/\bar{\mu})$. Because the cylinder is of periodic structure with period d, these equations hold from cell to cell. The wave numbers ξ and $\bar{\xi}$ have therefore to be analytically extended as periodic functions of period d, where γ is piecewise constant. Therefore, in principle, we have a second order differential equation with piecewise constant periodic coefficient.

The general theory of differential equations with periodic coefficients is due to Floquet[2], and an account of the theory can be found in Ince[3], Stoker[4], and Strutt[5]. In quantum-mechanical problems use of Floquet's theory was first made by Kronig and Penney[6], and a discussion of this problem can also be found in Brillouin[7]. More recently, Kaul and Herrmann[8], and Delph *et al.*[9, 10] have made use of Brillouin's procedure in solving some of the wave propagation problems in periodically structured elastic solid.

However, when the periodic coefficients are piecewise constant, Brillouin's method is unnecessarily long and arduous. Meissner has, however, shown that in this case an exact solution can be easily obtained in terms of elementary functions[11]. An excellent review of Meissner's equation and its various properties is contained in a recent paper by Hochstadt[12]. The chief advantage of the method is that the size of the characteristic determinant depends only on the order of the differential equation and is independent of the number of layers in the unit cell. Thus in the present problem, where we are dealing with a second order system, the order of the characteristic determinant is 2×2, irrespective of whether there are two or more layers in the cell.

The purpose of this paper is therefore twofold, (i) to illustrate use of Meissner's method, and (ii) to study in greater detail the problem of torsional waves in a hollow, structured cylinder. In addition to the discussion of dispersion spectrum, we also study the typical Liapounoff–Haupt arrangement of the characteristic sequence[13, 14], the problem of co-existence of periodic solutions, and the regions of stability and lability. Certain critical cases and properties of bandspectrum are not discussed, and can be found in an earlier report[15].

2. SOLUTION $R(r)$ IN THE RADIAL DIRECTION

To find the radial eigenfunctions $R_n(r)$, we first solve the boundary eigenvalue problem

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}R}{\mathrm{d}r}\right)+\frac{1}{r}(\beta^2r^2-1)R=0,\ \beta\equiv(\pi\kappa/2h),\ b<r<a,\, a>b$$

$$\frac{\mathrm{d}}{\mathrm{d}r}(rR)-2R=0 \text{ on } r=a \text{ and } r=b. \tag{7}$$

This is obviously a self-adjoint, singular boundary value problem with $\beta^2\geq 0$ and real, and the eigenfunctions are orthogonal with respect to weight r. The general solution is

$$R(r)=A\frac{2}{\beta}J_1(\beta r)-\frac{\pi}{2}B\beta Y_1(\beta r),\ \beta^2\geq 0 \tag{8}$$

where A and B are two arbitrary constants. Since the Wronskian $rW[J_1(\beta r), Y_1(\beta r)]=2/\pi(\neq 0)$, it follows that the solution is valid for all admissible values of separation constant β. In particular, when $\beta=0$, we get from eqn (8)

$$R(r)=Ar+B/r,\ \beta=0. \tag{9}$$

The two coefficients A and B are so chosen that the shear stress $\tau_{r\theta}$ vanishes on both boundaries ∂R: $r=a$ and $r=b$. Using the general solution $R(r)$ in the boundary condition $(7)_2$, and setting the determinant of the coefficients A and B equal to zero, we get the eigenvalue equation

$$[J_2(\beta a)Y_2(\beta b)-J_2(\beta b)Y_2(\beta a)]=0, \tag{10}$$

and the amplitude coefficient is given by

$$B:\ A=4J_2(\beta a):\ \pi\beta^2Y_2(\beta a). \tag{11}$$

Since the boundary value problem is self-adjoint, the eigenvalue equation (10) has *simple* roots, which are all real and discrete and can be arranged in an increasing order

$$0=\beta_0^2<\beta_1^2<\beta_2^2<\cdots<\beta_{n-1}^2<\beta_n^2<\cdots\ n\to\infty.$$

Corresponding to every eigenvalue there is an eigenfunction and these eigenfunctions form a *complete* set, which are mutually orthogonal with respect to weight r. For every eigenvalue β_n we can find B_n in terms of A_n, and therefore the radial eigenfunction $R_n(r)$ can be written in the form

$$R(r)=\frac{2A}{\beta Y_2(\beta a)}[Y_2(\beta a)J_1(\beta r)-J_2(\beta a)Y_1(\beta r)],\ \beta^2\geq 0 \tag{12}$$

where the suffix $n=0,1,2,\ldots$ belonging to $R(r)$, A and β has been suppressed for brevity. For $\beta_0=0$, the eigenfunction $R_0(r)$ takes the simple form

$$R_0(r)=A_0r, \tag{13}$$

and represents a pure torsional mode.

These eigenfunctions can be easily normalized, if we make use of the algebraic property that the product of two cylinder functions $C_1(r)$ and $D_1(r)$ can be written as

$$2rC_1(r)D_1(r)=\frac{\mathrm{d}}{\mathrm{d}r}[r^2(C_1(r)D_1(r)+C_0(r)D_0(r))-r(C_0(r)D_1(r)+C_1(r)\ D_0(r))]. \tag{14}$$

Making use of the normalizing condition

$$\int_b^a rR^2(r)\,\mathrm{d}r = 1,$$

and the algebraic identity just mentioned, we find that corresponding to the eigenfunction $R(r)$, the normalized coefficient A in eqn (12) is given by

$$A = \frac{1}{2}\frac{\beta Y_2(\beta a)}{\sqrt{(C(r)]_b{}^a)}}, \tag{15}$$

where

$$C(r) = \frac{1}{2} r^2[Y_2{}^2(\beta a)\{J_1{}^2(\beta r) - J_0(\beta r)J_2(\beta r)\} + J_2{}^2(\beta a)\{Y_1{}^2(\beta r) - Y_0(\beta r)Y_2(\beta r)\}$$
$$- J_2(\beta a)Y_2(\beta a)\{2J_1(\beta r)Y_1(\beta r) - J_0(\beta r)Y_2(\beta r) - Y_0(\beta r)J_2(\beta r)\}]. \tag{16}$$

The normalized set of orthogonal eigenfunctions are thus given by

$$R(r) = \frac{\beta Y_2(\beta a)}{\sqrt{(C(r)]_b{}^a)}}\left\{\frac{1}{\beta}J_1(\beta r) - \frac{J_2(\beta a)}{\beta Y_2(\beta a)}Y_1(\beta r)\right\},\ \beta^2 \ge 0. \tag{17}$$

In particular, in the limiting case when $\beta = 0$, we have a pure torsional mode

$$R_0(r) = \frac{2r}{\sqrt{(a^4 - b^4)}},\ a > b. \tag{18}$$

In the case of a solid cylinder $b = 0$, β_n are the simple roots of the eigenvalue equation $J_2(\beta a) = 0$, and the normalized set of orthogonal eigenfunctions take the simple form

$$R(r) = \frac{\sqrt{2}}{a}J_1(\beta r)/J_1{}^2(\beta a),\ \beta^2 \ge 0$$
$$R_0(r) = 2r/a^2,\ \beta^2 = 0. \tag{19}$$

If we write $b/a \equiv t$, then

$$\beta a \equiv \bar{\kappa} = \frac{\pi\kappa}{(1-t)},\ \beta b \equiv \bar{\kappa}t = \frac{\pi t\kappa}{(1-t)}$$

and the eigenvalue equation (10) takes the form

$$J_2(\bar{\kappa})Y_2(t\bar{\kappa}) - J_2(t\bar{\kappa})Y_2(\bar{\kappa}) = 0. \tag{20}$$

For $t < 1$, the asymptotic expansion for the pth root of this transcendental equation is given by the McMahon series [16],

$$\kappa_p \approx p[1 + 30tq^2\{1 - 3q^2(1 + 11t + t^2) - 18q^4(11 - 9t - 109t^2 - 9t^3 + 11t^4) + \cdots\}], \tag{21}$$

where

$$q \equiv \frac{(1-t)}{4\pi pt},\ p = 1, 2, 3, \ldots \tag{22}$$

It is obvious that the lowest root of eqn (20) is $\kappa_0 = 0$. The higher roots of this equation are given with sufficient accuracy by the asymptotic formula (21). For $t = 2/3$, the first three roots are $\kappa_1 = 1.030\ 4056$, $\kappa_2 = 2.015\ 6684$, $\kappa_3 = 3.010\ 5057$. For $p > 3$ the higher roots become increasingly accurate. For $t = 1/3$, $\kappa_2 = 2.115\ 5496$, $\kappa_3 = 3.081\ 0800$, $\kappa_4 = 4.061\ 9010$, etc.

3. DISPLACEMENT FIELD $Z(z)$ IN ONE CELL

The domain of a typical cell $R_1 \times R_d$ is the union of two layers $R_1 \times R_l$ and $R_1 \times R_{\bar{l}}$ with an interface $R_l \cap R_{\bar{l}}$. The displacement components $Z(z) \in R_l$ and $\bar{Z}(z) \in R_{\bar{l}}$ are governed by the ordinary differential equations (6)

$$
\begin{gathered}
Z''(z) + \alpha^2 Z(z) = 0, \ \forall z \in R_l, \ R_l: 0 < z < l \\
\bar{Z}''(z) + \bar{\alpha}^2 \bar{Z}(z) = 0, \ \forall z \in R_{\bar{l}}, \ R_{\bar{l}}: l < z < d, \\
d = (l + \bar{l})
\end{gathered} \tag{23}
$$

where

$$
\begin{gathered}
\alpha = \frac{\pi}{2h} \xi, \ \bar{\alpha} = \frac{\pi}{2h} \bar{\xi}, \\
\xi^2 = (\Omega^2 - \kappa^2), \ \bar{\xi}^2 = ((\gamma\Omega)^2 - \kappa^2), \\
\gamma^2 = (c/\bar{c})^2, \ c^2 = \mu/\rho, \ \bar{c}^2 = \bar{\mu}/\bar{\rho}.
\end{gathered} \tag{24}
$$

The continuity of displacement $u(r, z)$ and shear stress $\tau_{z\theta}$ at the interface $R_l \cap R_{\bar{l}}$ requires that the functions $Z(z)$ and $\bar{Z}(z)$ must satisfy the continuity conditions

$$
\begin{gathered}
Z(l) = \bar{Z}(l), \\
\mu Z'(l) = \bar{\mu} \bar{Z}'(l),
\end{gathered} \tag{25}
$$

where accent denotes differentiation with respect to the coordinate z.

We assume the general solution of the differential equation (23) in the form

$$
\begin{gathered}
Z(z) = A \cos \alpha z + B \frac{1}{\alpha} \sin \alpha z, \ \alpha^2 \geq 0, \ z \in R_l \\
\bar{Z}(z) = \bar{A} \cos \bar{\alpha} z + \bar{B} \frac{1}{\bar{\alpha}} \sin \bar{\alpha} z, \ \bar{\alpha}^2 \geq 0, \ z \in R_{\bar{l}}.
\end{gathered} \tag{26}
$$

The two linearly independent solutions

$$
w_1(z) \equiv \cos \alpha z, \ w_2(z) \equiv \frac{1}{\alpha} \sin \alpha z \tag{27}
$$

satisfy the initial conditions

$$
w_1(0) = 1, \ w_1'(0) = 0, \ w_2(0) = 0, \ w_2'(0) = 1 \tag{28}
$$

and therefore the Wronskian $W[w_1(z), w_2(z)] = 1$. This establishes the linear independence for $\alpha^2 \geq 0$. Similarly it follows that $\cos \bar{\alpha} z$ and $(\sin \bar{\alpha} z)/\bar{\alpha}$ are also linearly independent for $\bar{\alpha}^2 \geq 0$.

Using the continuity equation (25) at the interface $z = l$, we can determine the coefficients $\bar{A}$ and $\bar{B}$ in terms of the coefficients A and B. These coefficients are

$$
\begin{gathered}
\bar{A} = A\left(C\bar{C} + \frac{\mu\alpha}{\bar{\mu}\bar{\alpha}} S\bar{S}\right) + \frac{B}{\alpha}\left(S\bar{C} - \frac{\mu\alpha}{\bar{\mu}\bar{\alpha}} C\bar{S}\right), \\
\frac{\bar{B}}{\bar{\alpha}} = A\left(C\bar{S} - \frac{\mu\alpha}{\bar{\mu}\bar{\alpha}} S\bar{C}\right) + \frac{B}{\alpha}\left(S\bar{S} + \frac{\mu\alpha}{\bar{\mu}\bar{\alpha}} C\bar{C}\right),
\end{gathered} \tag{29}
$$

where

$$
C \equiv \cos \alpha l, \ \bar{C} \equiv \cos \bar{\alpha} l, \ S \equiv \sin \alpha l, \ \bar{S} \equiv \sin \bar{\alpha} l \tag{30}
$$

and the coefficients A and B are still to be determined.

Having determined the coefficients $\bar{A}$ and $\bar{B}$, the general solution in the fundamental cell $R_l \cup R_{\bar{l}}$ in terms of the coefficients A and B takes the form

$$Z(z) = Aw_1(z) + Bw_2(z),\ \alpha^2 \geq 0,\ z \in R_l$$
$$\bar{Z}(z) = A\bar{w}_1(z) + B\bar{w}_2(z),\ \bar{\alpha}^2 \geq 0,\ z \in R_{\bar{l}} \tag{31}$$

where now

$$w_1(z) \equiv \cos \alpha z,\ w_2(z) \equiv \frac{1}{\alpha} \sin \alpha z,$$
$$\bar{w}_1(z) \equiv C \cos \bar{\alpha}(z-l) - \frac{\mu\alpha}{\bar{\mu}\bar{\alpha}} S \sin \bar{\alpha}(z-l), \tag{32}$$
$$\bar{w}_2(z) \equiv \frac{1}{\alpha} S \cos \bar{\alpha}(z-l) + \frac{\mu}{\bar{\mu}\bar{\alpha}} C \sin \bar{\alpha}(z-l),$$

and the respective Wronskians are

$$W[w_1, w_2] = 1,\ W[\bar{w}_1, \bar{w}_2] = \mu/\bar{\mu}. \tag{33}$$

In order that $\bar{w}_1$ and $\bar{w}_2$ be linearly independent, their corresponding Wronskian must be non-zero. This requires that $0 < \mu/\bar{\mu}$.

It can be easily verified that at the interface $z = l$ of the two layers in a primitive cell

$$w_1(l) = \bar{w}_1(l),\ w_2(l) = \bar{w}_2(l)$$
$$\mu w_1'(l) = \bar{\mu}\bar{w}_1'(l),\ \mu w_2'(l) = \bar{\mu}\bar{w}_2'(l) \tag{34}$$

and therefore the general solution (31) satisfies the continuity conditions (25).

It may be finally remarked that this process can be easily generalized if there are more than two layers in a primitive cell. Such an extension to three and four layered composite is contained in Meissner's paper[11].

4. QUASI-PERIODIC SOLUTIONS IN A CYLINDER WITH PERIODIC STRUCTURE

In the case of a cylinder with periodic structure with period d, we require that at the common interface between two adjacent cells, the displacement and stress be continuous and at least quasi-periodic. The conditions of continuity at the cell interface $z = d$ are

$$\bar{u}(r, d^-) = u(r, d^+),$$
$$\bar{\tau}_{z\theta}(r, d^-) = \tau_{z\theta}(r, d^+). \tag{35}$$

Quasi-periodicity from cell to cell requires that

$$u(r, z+d) = \sigma u(r, z),$$
$$\tau_{z\theta}(r, z+d) = \sigma\tau_{z\theta}(r, z),\ 0 < z < l \tag{36}$$

$$\bar{u}(r, z+d) = \sigma\bar{u}(r, z),$$
$$\bar{\tau}_{z\theta}(r, z+d) = \sigma\bar{\tau}_{z\theta}(r, z),\quad l < z < d \tag{37}$$

where σ is a suitable constant to be determined[3]. Combining the continuity and the quasi-periodic conditions we find that at the cell interface the appropriate continuity conditions are

$$\bar{u}(r, d^-) = \sigma u(r, 0^+),$$
$$\bar{\tau}_{z\theta}(r, d^-) = \sigma\tau_{z\theta}(r, 0^+). \tag{38}$$

In each lamina of the unit cell, the tangential component of the displacement can be expressed in the product form

$$\bar{u}(r, z) = R(r)\bar{Z}(z), \quad u(r, z) = R(r)Z(z) \tag{39}$$

where the eigenfunctions $R(r)$, $Z(z)$ and $\bar{Z}(z)$ are defined by eqns (17) and (31), respectively. The continuity conditions (38) at a typical cell interface therefore takes the simple form

$$\begin{aligned} \bar{Z}(d^-) &= \sigma Z(0^+), \\ \bar{\mu}\bar{Z}'(d^-) &= \sigma\mu Z'(0^+). \end{aligned} \tag{40}$$

The functions Z and $\bar{Z}$ contain two arbitrary constants A and B which can now be determined if we make use of the continuity equations (40). Substituting the functions Z and $\bar{Z}$ in these two equations, we get a pair of linear homogeneous equations

$$\begin{bmatrix} \bar{w}_1(d) - \sigma & \bar{w}_2(d) \\ \bar{\mu}\bar{w}_1'(d) & \bar{\mu}\bar{w}_2'(d) - \sigma\mu \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = 0. \tag{41†}$$

Existence of a non-trivial solution for a system of homogeneous equations requires that the determinant of the coefficients A and B must vanish. This gives us the characteristic σ-equation

$$\sigma^2 - \sigma\left[\bar{w}_1(d) + \frac{\bar{\mu}}{\mu}\bar{w}_2'(d)\right] + 1 = 0, \tag{42}$$

where we have used the fact that $W[\bar{w}_1(d), \bar{w}_2(d)] = \bar{\mu}/\mu$. In addition, for each value of the constant σ, the amplitude ratio is given by

$$A:B = \bar{w}_2(d):[\sigma - \bar{w}_1(d)] = [\sigma\mu - \bar{\mu}\bar{w}_2'(d)]:\bar{\mu}\bar{w}_1'(d). \tag{43}$$

Assuming $\sigma \neq 0$, we can rewrite the characteristic equation (42) as

$$\sigma + \frac{1}{\sigma} = H, \tag{44}$$

where

$$H \equiv \bar{w}_1(d) + \frac{\bar{\mu}}{\mu}\bar{w}_2'(d). \tag{45}$$

It easily follows that

$$\left(\sqrt{\sigma} + \frac{1}{\sqrt{\sigma}}\right)^2 = H + 2, \quad \left(\sqrt{\sigma} - \frac{1}{\sqrt{\sigma}}\right)^2 = H - 2, \tag{46}$$

and therefore

$$\sigma - \frac{1}{\sigma} = \sqrt{[(H-2)(H+2)]}. \tag{47}$$

If σ_1 and σ_2 are two roots of the eqn (42), then it is obvious that $\sigma_1\sigma_2 = 1$ and $(\sigma_1 + \sigma_2) = H$. Evidently $\sigma_1 \equiv \sigma$ and $\sigma_2 \equiv 1/\sigma$ are the two roots and are explicitly given by

$$\begin{matrix} \sigma_1 \equiv \sigma \\ \sigma_2 \equiv 1/\sigma \end{matrix} = \frac{1}{2}[H \pm \sqrt{(H^2 - 4)}] = \frac{1}{4}[\sqrt{(H+2)} \pm \sqrt{(H-2)}]^2. \tag{48}$$

†Note that the 2 × 2-form of the determinant remains unchanged if there are more than two layers in a primitive cell.

We now distinguish three cases:

(*i*) $|H| < 2$. If $|H| < 2$, then from (48) it follows that σ_1 and σ_2 are complex conjugate and of absolute value unity. In this case all solutions are bounded since quasi-periodicity from cell to cell requires that after n cells

$$Z(z + nd) = \sigma^n Z(z), \quad \bar{Z}(z + nd) = \sigma^n \bar{Z}(z) \tag{49}$$

where now $|\sigma| = 1$.

(ii) $|H| > 2$. If $|H| > 2$, then from (48) it follows that $\sigma_1 \neq \sigma_2$ and the two roots are real. Since $\sigma_1\sigma_2 = 1$, it implies that if one root is greater than unity then the other must be less than unity in absolute value. Thus as the number of cells $n \to \infty$, one of the solutions will become unbounded and the other will decay to zero.

(iii) $H = \pm 2$. If $H = \pm 2$, then $\sigma_1 = \sigma_2 = \sigma$ is a root of multiplicity 2. Since $\sigma_1\sigma_2 = 1$, it follows that $\sigma^2 = 1$ and therefore either $\sigma = +1$ or $\sigma = -1$, each being a double root. When $\sigma = 1$ for $H = 2$, the solution is periodic with period d, since in this case $Z(z + d) = Z(z)$. When $\sigma = -1$ for $H = -2$, the solution is periodic with period $2d$, because in this case $Z(z + 2d) = (-1)^2 Z(z) = Z(z)$. In the case of double roots only one solution is basically periodic, the second linearly independent solution is in general aperiodic [3]†

The characteristic equation (42), or equivalently eqn (46) takes a familiar form if we introduce the transformation

$$\sigma = \exp 2i(\tau + \bar{\tau})\lambda \tag{50}$$

where i is a primitive fourth root of unity,

$$\tau \equiv \frac{\pi l}{4h}, \quad \bar{\tau} \equiv \frac{\pi \bar{l}}{4h} \tag{51}$$

and λ is a characteristic exponent (also known as Floquet's exponent), which determines the phase shift. In the transformation (50), λ is congruent modulo $\pi/(\tau + \bar{\tau})$. For a given σ, the value of λ is determined uniquely in the interval $I[\lambda]$: $[0, \pi/(\tau + \bar{\tau})]$. Having determined λ in the interval $I[\lambda]$, all other values of λ are congruent to it modulo $\pi/(\tau + \bar{\tau})$.

5. THE CHARACTERISTIC λ-EQUATION

In terms of the characteristic exponent λ, the two characteristic equations (46) can be written in the useful form

$$\sin^2(\tau + \bar{\tau})\lambda = \frac{1}{4}(2 - H),$$
$$\cos^2(\tau + \bar{\tau})\lambda = \frac{1}{4}(2 + H). \tag{52}$$

The function H is defined in terms of $\bar{w}_1(d)$ and $\bar{w}_2'(d)$ which can be easily obtained from $\bar{w}_1(z)$, $\bar{w}_2(z)$, which are given by (32). After some algebraic simplification it can be shown that the two characteristic equations (52), take the interesting product form

$$\sin^2(\tau + \bar{\tau})\lambda = \left\{C(\tau\xi)S(\bar{\tau}\bar{\xi}) + \frac{\bar{\mu}\bar{\xi}}{\mu\xi} S(\tau\xi)C(\bar{\tau}\bar{\xi})\right\}\left\{C(\tau\xi)S(\bar{\tau}\bar{\xi}) + \frac{\mu\xi}{\bar{\mu}\bar{\xi}} S(\tau\xi)C(\bar{\tau}\bar{\xi})\right\},$$
$$\cos^2(\tau + \bar{\tau})\lambda = \left\{C(\tau\xi)C(\bar{\tau}\bar{\xi}) - \frac{\bar{\mu}\bar{\xi}}{\mu\xi} S(\tau\xi)\bar{S}(\bar{\tau}\bar{\xi})\right\}\left\{C(\tau\xi)C(\bar{\tau}\bar{\xi}) - \frac{\mu\xi}{\bar{\mu}\bar{\xi}} S(\tau\xi)S(\bar{\tau}\bar{\xi})\right\}, \tag{53}$$

†This is reminiscent of a similar situation in the theory of ordinary differential equations, when the indicial equation has roots of higher multiplicity.

where S and C are abbreviations for trigonometric sine and cosine functions, respectively. We may now note that the two trigonometric functions $\sin^2(\tau+\bar{\tau})\lambda$ and $\cos^2(\tau+\bar{\tau})\lambda$, are even-functions about $\lambda=0$ and $\lambda=\pi/[2(\tau+\bar{\tau})]$. In addition, λ is congruent modulo $\pi/(\tau+\bar{\tau})$. It is therefore sufficient to restrict Floquet's wave number λ to the interval $I_B[\lambda]$: $[0, \pi/2(\tau+\bar{\tau})]$. With this restriction on the value of λ, the frequency spectrum Ω vs λ has a *zone* structure, and in the first Brillouin zone I_B, λ varies from 0 to $\pi/[2(\tau+\bar{\tau})]$[7].

The characteristic roots Ω of this characteristic equation can be arranged in an increasing order and thus can be indexed by a suffix n. It can be shown graphically that the order follows the well-known Liapounoff–Haupt sequence[13, 14]. In particular, it can be exhibited explicitly at the end-points of the Brillouin zone, $\lambda=0$ and $\lambda=\pi/[2(\tau+\bar{\tau})]$.

Of particular interest is the case when $H=\pm 2$, that is, when $\sigma=\pm 1$. When $\sigma=1$, $\lambda=n\pi/(\tau+\bar{\tau})$, $n=0,1,2,\dots$ and in the *reduced zone scheme* $\lambda=0$ corresponds to the left end-point of the interval I_B. Since $\sigma=1$ is a zero of multiplicity 2, out of the two linearly independent solutions only one is periodic with period d; the second solution is aperiodic. Corresponding to the periodic solutions with period d, the characteristic equation (53), uncouples into two equations

$$\begin{aligned}
&(1) \qquad \tan\bar{\tau}\bar{\xi}+\frac{\bar{\mu}\bar{\xi}}{\mu\xi}\tan\tau\xi=0,\\
& \hspace{6cm} \sigma=1, H=2\\
&(2) \qquad \tan\bar{\tau}\bar{\xi}+\frac{\mu\xi}{\bar{\mu}\bar{\xi}}\tan\tau\xi=0.
\end{aligned} \tag{54}$$

When $\sigma=-1$, $\lambda=(n+1/2)\pi/(\tau+\bar{\tau})$, $n=0,1,2,\dots$ and in the reduced zone scheme $\lambda=\pi/[2(\tau+\bar{\tau})]$ corresponds to the right end-point of the interval I_B. Since $\sigma=-1$ is a zero of multiplicity 2, only one linearly independent solution is periodic with period 2d. Corresponding to the periodic solution with period $2d$, the characteristic equation $(53)_2$ uncouples into two equations

$$\begin{aligned}
&(1) \qquad \cot\bar{\tau}\bar{\xi}-\frac{\bar{\mu}\bar{\xi}}{\mu\xi}\tan\tau\xi=0,\\
& \hspace{6cm} \sigma=-1,\ H=-2\\
&(2) \qquad \cot\bar{\tau}\bar{\xi}-\frac{\mu\xi}{\bar{\mu}\bar{\xi}}\tan\tau\xi=0.
\end{aligned} \tag{55}$$

One can easily see that the multiplicity of the root $\sigma=1$, corresponds to the multiplicity of the root $\lambda=n\pi/(\tau+\bar{\tau})$, and multiplicity of the root $\sigma=-1$ corresponds to the multiplicity of the root $\lambda=(n+1/2)\pi/(\tau+\bar{\tau})$. This follows from the fact that in the first case $\sin^2(\tau+\bar{\tau})\lambda$ and its first derivative both vanish at the left end-point of the Brillouin zone. In the second case $\cos^2(\tau+\bar{\tau})\lambda$ and its first derivative both vanish at the right end-point of the Brillouin zone.

If $Z(z)$ is a solution then $Z(-z)$ is also a solution because the differential equation is invariant under coordinate reflection. These two solutions are in general linearly independent, unless $Z(z)$ is an even or an odd solution. However, we have shown the existence of at least one periodic solution for $\sigma=\pm 1$. Therefore, either $Z(z)$ is a periodic solution or it is not. If $Z(z)$ is a periodic solution then it is either even or odd. If it is not even or odd then we can construct the solutions $[Z(z)+Z(-z)]$ and $[Z(z)-Z(-z)$ which are even-periodic and odd-periodic, respectively. In either case, even-periodic and odd-periodic solutions can always be determined, when periodic solutions exist. Thus at the left (right) end-point of the Brillouin zone, even- and odd-periodic solutions with period $d(2d)$ can always be determined.

The dispersion spectra on an extended zone scheme are shown in Fig. 1 for $\kappa_0=0$, and in Fig. 2 for $\kappa_1=1.18920$. The material parameters are $t=1/3$, $\gamma=1/4$, $\bar{\mu}/\mu=40$, $\bar{\rho}/\rho=5/2$ and $\bar{\tau}/\tau=5$. Real segments of the spectra are shown as full lines, imaginary segments are shown as dotted lines. In Fig. 2, point A corresponds to the limiting case $\xi=0$, and point B to the limiting case when $\bar{\xi}=0$. In these figures, hatched regions represent stopping bands.

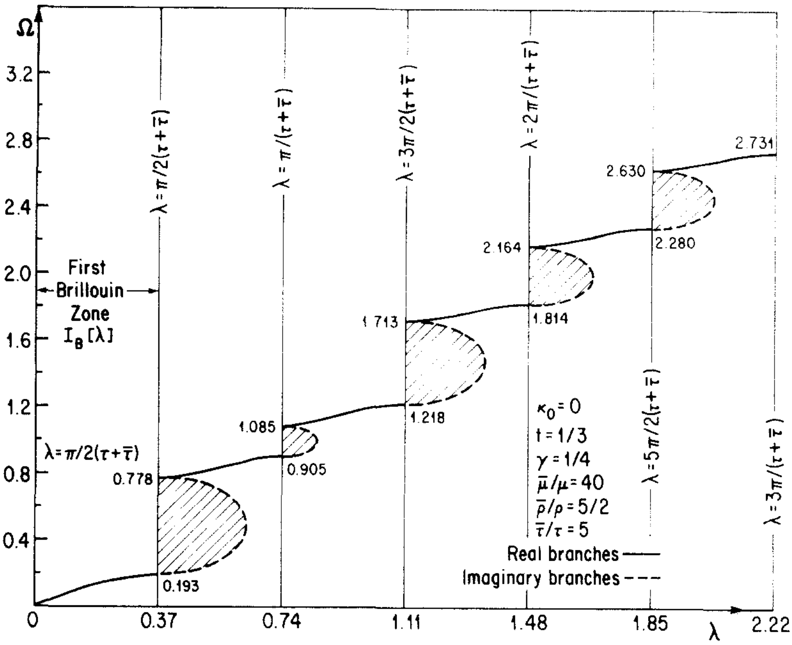

Fig. 1. Floquet's spectrum on an extended zone scheme for pure torsional modes $\kappa_0 = 0$. Real segments are shown as full lines, imaginary segments are shown as dotted lines. Hatched regions represent stopping bands. Brillouin zone $I_B[\lambda]$: $[0, \pi/2(\tau+\bar{\tau})]$.

6. CHARACTERISTIC λ-EQUATION FOR $\kappa_0 = 0$

In the case of pure torsional mode $\beta_0 = 0$ and this implies $\kappa_0 = 0$, $\xi = \Omega$, $\bar{\xi} = \gamma\Omega$. The characteristic equations (53) now take a simple form

$$\sin^2(\tau+\bar{\tau})\lambda = \left\{C(\tau\Omega)S(\gamma\bar{\tau}\Omega) + \frac{\bar{\mu}\gamma}{\mu}S(\tau\Omega)C(\gamma\bar{\tau}\Omega)\right\}\left\{C(\tau\Omega)S(\gamma\bar{\tau}\Omega) + \frac{\mu}{\bar{\mu}\gamma}S(\tau\Omega)C(\gamma\bar{\tau}\Omega)\right\},$$

$$\cos^2(\tau+\bar{\tau})\lambda = \left\{C(\tau\Omega)C(\gamma\bar{\tau}\Omega) - \frac{\bar{\mu}\gamma}{\mu}S(\tau\Omega)S(\gamma\bar{\tau}\Omega)\right\}\left\{C(\tau\Omega)C(\gamma\bar{\tau}\Omega) - \frac{\mu}{\bar{\mu}\gamma}S(\tau\Omega)S(\gamma\bar{\tau}\Omega). \qquad (56)$$

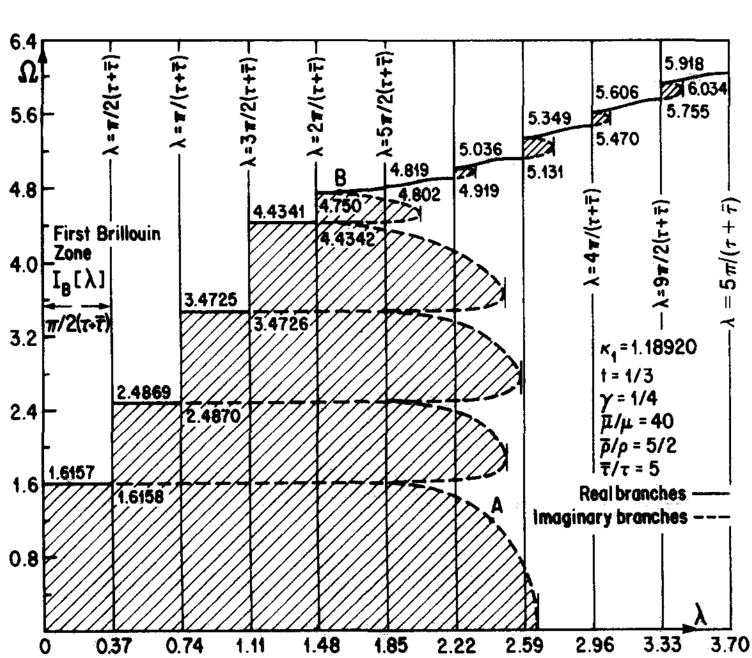

Fig. 2. Floquet's spectrum on an extended zone scheme for torsional modes with one radial node, $\kappa_1 = 1.1892$. Real segments are shown as full lines, imaginary segments as dotted lines. Hatched regions represent stopping bands. Wave number $\xi = 0$ at point A, and at point B, $\bar{\xi} = 0$. Brillouin zone $I_B[\lambda]$: $[0, \pi/2(\tau+\bar{\tau})]$.

At the end-points of the Brillouin zone, for $\sigma = 1$ the characteristic equations are

$$\begin{aligned} &(1) \quad \tan \gamma\bar{\tau}\Omega + \frac{\gamma\bar{\mu}}{\mu} \tan \tau\Omega = 0, \\ &(2) \quad \tan \gamma\bar{\tau}\Omega + \frac{\mu}{\gamma\bar{\mu}} \tan \tau\Omega = 0, \end{aligned} \qquad \sigma = 1,\ H = 2 \tag{57}$$

and for $\sigma = -1$, the characteristic equations are

$$\begin{aligned} &(\hat{1}) \quad \cot \gamma\bar{\tau}\Omega - \frac{\gamma\bar{\mu}}{\mu} \tan \tau\Omega = 0, \\ &(\hat{2}) \quad \cot \gamma\bar{\tau}\Omega - \frac{\mu}{\gamma\bar{\mu}} \tan \tau\Omega = 0. \end{aligned} \qquad \sigma = -1,\ H = -2 \tag{58}$$

7. LIAPOUNOFF–HAUPT SEQUENCE

We now demonstrate graphically that at the end-points of the Brillouin zone the characteristic values of λ can be arranged in a Liapounoff–Haupt sequence, [13, 14]. This is an interesting property of certain differential equations with periodic coefficients, and in this relatively simple case it can be demonstrated explicitly. We first consider the case of pure torsional mode, corresponding to $\kappa_0 = 0$.

Let $\gamma\bar{\mu}/\mu \equiv \Gamma$, $\gamma\bar{\tau}/\tau \equiv T$ and $\tau\Omega \equiv x$. Then for $\sigma = 1$, eqn (57) can be written as

$$\begin{aligned} &(1) \quad Tx = k\pi - \tan^{-1} \Gamma \tan x, \\ &(2) \quad Tx = k\pi - \tan^{-1} \frac{1}{\Gamma} \tan x, \end{aligned} \tag{59}$$

and for $\sigma = -1$, eqn (58) takes the form

$$\begin{aligned} &(\hat{1}) \quad Tx = (2k+1)\pi/2 - \tan^{-1} \Gamma \tan x, \\ &(\hat{2}) \quad Tx = (2k+1)\pi/2 - \tan^{-1} \frac{1}{\Gamma} \tan x, \end{aligned} \tag{60}$$

where in these equations $k = 0, 1, 2, \ldots$.

First consider the multi-valued function

$$f(x) = k\pi - \tan^{-1} \Gamma \tan x, \quad k = 0, 1, 2, \ldots \tag{61}$$

for a fixed value of parameter Γ. The slope of the function is

$$f'(x) = -\Gamma/(\cos^2 x + \Gamma^2 \sin^2 x), \tag{62}$$

and its curvature is

$$f'' = \Gamma(\Gamma^2 - 1) \frac{\sin 2x}{(\cos^2 x + \Gamma^2 \sin^2 x)^2}. \tag{63}$$

Since $\Gamma > 0$, the slope is always negative and reaches a value $-\Gamma$ for $x = 0, \pi, 2\pi, \ldots$, and a value of $-1/\Gamma$ for $x = \pi/2, 3\pi/2, 5\pi/2, \ldots$. For $\Gamma = 1$, the curvature is zero. For $\Gamma \neq 1$, the curvature is zero when $x = 0, \pi/2, \pi, 3\pi/2, \ldots$. When $\Gamma = 1$, $f(x) = k\pi - x$, and represents a straight line with slope -1 and zero curvature. Further $f(x) = k\pi$ for $x = 0, \pi, 2\pi, \ldots$ and for $x = \pi/2, 3\pi/2, 5\pi/2, \ldots$, $f(x) = (k - (1/2))\pi$. For other values of x, the function can be readily computed and the spectral lines are shown in Fig. 3, marked 1. We can similarly analyze the function

$$f(x) = k\pi - \tan^{-1} \frac{1}{\Gamma} \tan x,$$

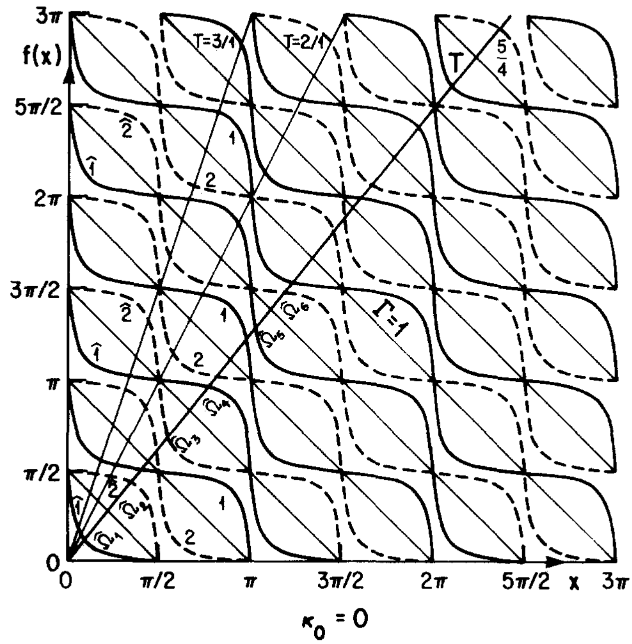

Fig. 3. Plot of the multi-valued function $f(x)$ vs x for $\kappa_0 = 0$. Straight lines with slope-1 correspond to the case $\Gamma = 1$. For $T = 3/1$, $\Omega_3 = \Omega_4$, $\Omega_7 = \Omega_8$, etc. for $T = 2/1$, $\hat{\Omega}_3 = \hat{\Omega}_4$, $\Omega_5 = \Omega_6$, $\hat{\Omega}_9 = \hat{\Omega}_{10}$, etc. and for $T = 5/4$, $\hat{\Omega}_9 = \hat{\Omega}_{10}$, etc.

and the spectral lines are shown in Fig. 3, marked 2. The points of intersection of the set of these two families of curves with the straight line with slope T locates the zeros of the two characteristic equations $(57)_{1,2}$. It is obvious from the figure that the zeros of these two characteristic equations, which correspond to periodic solutions with period d, form a sub-sequence

$$0 = \Omega_0 < \Omega_1 \leq \Omega_2 < \Omega_3 \leq \Omega_4 < \Omega_5 \leq \Omega_6 < \cdots \tag{64}$$

We can similarly plot the two functions $f(x)$ corresponding to the case $\sigma = -1$. In Fig. (3), these curves are marked $\hat{1}$ and $\hat{2}$. The intersection of these curves with the straight line with slope T, locates the zeros of the two characteristic equations (58). If $\hat{\Omega}$ represents an element of this sequence, then it is obvious from the figure that the elements can be ordered in a sub-sequence

$$\hat{\Omega}_1 \leq \hat{\Omega}_2 < \hat{\Omega}_3 \leq \hat{\Omega}_4 < \hat{\Omega}_5 \leq \hat{\Omega}_6 < \cdots \tag{65}$$

Combining the two sub-sequences, we have the Liapounoff–Haupt sequence

$$0 = \Omega_0 < \hat{\Omega}_1 \leq \hat{\Omega}_2 < \Omega_1 \leq \Omega_2 < \hat{\Omega}_3 \leq \hat{\Omega}_4 < \Omega_3 \leq \Omega_4 < \cdots \tag{66}$$

It may be remarked that such a sequence is a characteristic of a wide class of problems governed by certain differential equations with periodic coefficients.

We now consider the general case for the radial eigenvalue $\kappa^2 > 0$. In this case the characteristic eigenvalues at the end-points of the Brillouin zone, for $\sigma = +1$ and $\sigma = -1$ are given by

(1) $$\bar{\tau}\bar{\xi} = k\pi - \tan^{-1}\frac{\bar{\mu}\bar{\xi}}{\mu\xi}\tan\tau\xi,$$

(2) $$\bar{\tau}\bar{\xi} = k\pi - \tan^{-1}\frac{\mu\xi}{\bar{\mu}\bar{\xi}}\tan\tau\xi;$$

$(\hat{1})$ $$\bar{\tau}\bar{\xi} = (2k+1)\frac{\pi}{2} - \tan^{-1}\frac{\bar{\mu}\bar{\xi}}{\mu\xi}\tan\tau\xi,$$

$(\hat{2})$ $$\bar{\tau}\bar{\xi} = (2k+1)\frac{\pi}{2} - \tan^{-1}\frac{\mu\xi}{\bar{\mu}\bar{\xi}}\tan\tau\xi. \tag{67}$$

Consider the first of the four equations (67). In the notation defined earlier, this equation can be rewritten as

$$T\surd[x^2-(\kappa\tau/\gamma)^2]=k\pi+f_1(x), \tag{68}$$

where the function $f_1(x)$ is defined as

$$\begin{aligned}
f_1(x) &= -\tan^{-1}\left(\Gamma\sqrt{\left(\frac{x^2-(\kappa\tau/\gamma)^2}{x^2-(\kappa\tau)^2}\right)}\tan\surd[x^2-(\kappa\tau)^2]\right),\ x>\kappa\tau/\gamma\\
&=0,\ x=\kappa\tau/\gamma\\
&=-i\tanh^{-1}\left(\Gamma\sqrt{\frac{(\kappa\tau/\gamma)^2-x^2}{x^2-(\kappa\tau)^2}}\tan\surd[x^2-(\kappa\tau)^2],\ \kappa\tau<x<\kappa\tau/\gamma\right.\\
&=0,\ x=\surd[(\kappa\tau)^2+\pi^2]\\
&=0,\ x=\surd[(\kappa\tau)^2+4\pi^2]\\
&=-i\tanh^{-1}[\Gamma\kappa\tau\surd[(1/\gamma)^2-1],\ x=\kappa\tau,\ \gamma<1\\
&=-i\tanh^{-1}\left(\Gamma\sqrt{\left(\frac{(\kappa\tau/\gamma)^2-x^2}{(\kappa\tau)^2-x^2}\right)}\tanh\surd[(\kappa\tau)^2-x^2],\ 0<x<\kappa\tau\right.\\
&=-i\tanh^{-1}\left(\frac{\Gamma}{\gamma}\tanh\kappa\tau\right),\ x=0.
\end{aligned}$$

For $t\equiv b/a=1/3$, the first radial eigenvalue is $\kappa_1=1.18920$. For the purpose of numerical computations we select as before

$$\bar{\mu}/\mu=40,\ \bar{\rho}/\rho=5/2,\ l/a=3/5,\ \bar{l}/a=3,$$

which leads to

$$\gamma=1/4,\ \Gamma=10,\ \tau=9\pi/20,\ \bar{\tau}=9\pi/4,\ T=5/4,$$

as the values of the parameters. We now plot the function $k\pi+f_1(x)$, for $k=0,1,2,\ldots$. The points of intersection of these curves with the parabola $T\surd[x^2-(\kappa\tau/\gamma)^2]$, locate the zeros of the characteristic equation $(67)_1$. In Fig. 4, the parabola and the spectral lines are drawn for $x\geqslant\kappa\tau/\gamma$, and the spectral lines corresponding to the characteristic equation $(67)_1$ are marked 1. The sepctral lines for the second characteristic equation are also shown in the same figure, and marked 2. It is obvious from the figure that the zeros of the two characteristic equations, when arranged in an increasing order, form a sub-sequence, which is similar to the case $\kappa_0=0$. The second system of characteristic equations are plotted similarly and are shown in the same figure. These spectral lines are marked $\hat{1}$ and $\hat{2}$, respectively. Again it is obvious from the figure that there exists a sub-sequence, similar to the case $\kappa_0=0$. Combining the two sub-sequences, we find that the zeros of these four equations when arranged in an ascending order form the Liapounoff–Haupt sequence.

8. CO-EXISTENCE OF PERIODIC SOLUTIONS

To study the problem of co-existence of periodic solutions, we first consider the case of pure torsional mode when $\kappa_0=0$. For $\sigma=1$, the two characteristic equations are

$$\begin{aligned}
&\Gamma\sin x\cos Tx+\cos x\sin Tx=0,\\
&\Gamma\cos x\sin Tx+\sin x\cos Tx=0.
\end{aligned} \tag{69}$$

Consider first the case when $\Gamma=1$. In Fig. 3, the spectral lines corresponding to $\Gamma=1$ are straight lines with slope -1. The points of intersection of these lines with a line of slope T are all roots of multiplicity 2. This also follows from the fact that when $\Gamma=1$, these two equations are equivalent to one equation $\cos 2(1+T)x=1$, whose zeros are given by $x_n=n\pi/(1+T)$,

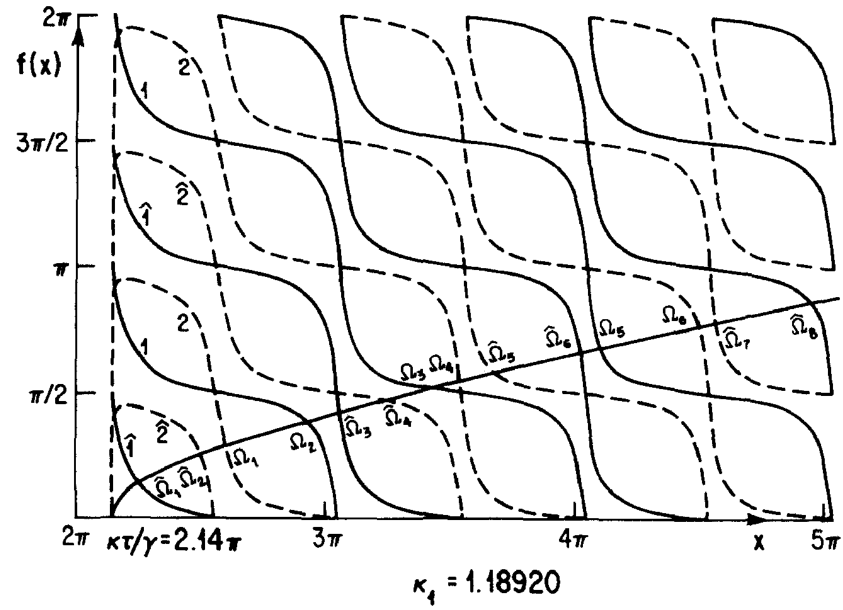

Fig. 4. Plot of the multi-valued functions $f(x)$ vs x for $\kappa_1 = 1.1892$, and for $x \geqslant \kappa\tau/\gamma = 2.14056\pi$. A typical parabolic curve $T\sqrt{[x^2-(\kappa\tau/\gamma)^2]}$ is also shown intersecting the multivalued function $f(x)$. $T = \gamma\bar{\tau}/\tau = 5/4$.

$n = 0, 1, 2, \ldots$. These are zeros of multiplicity 2, because the derivative $\sin 2(1+T)x_n \equiv 0$. Thus, when $\sigma = 1$ and $\Gamma = 1$, we arrive at the Ω-sub-sequence

$$0 = \Omega_0 < \Omega_1 = \Omega_2 < \Omega_3 = \Omega_4 < \Omega_5 = \Omega_6 < \cdots .$$

Similarly when $\sigma = -1$, the two characteristic equations are

$$\begin{aligned} &\Gamma \sin x \sin Tx - \cos x \cos Tx = 0, \\ &\Gamma \cos x \cos Tx - \sin x \sin Tx = 0. \end{aligned} \tag{70}$$

When $\Gamma = 1$, these two equations are equivalent to one equation $\cos 2(1+T)x = -1$, whose zeros are $x_n = (n + 1/2)\pi/(1+T)$, $n = 0, 1, 2, \ldots$. Again these are zeros of multiplicity 2, because the derivative $\sin 2(1+T)x_n \equiv 0$. Thus in this case we arrive at the Ω-sub-sequence

$$\hat{\Omega}_1 = \hat{\Omega}_2 < \hat{\Omega}_3 = \hat{\Omega}_4 < \hat{\Omega}_5 = \Omega_6 < \cdots .$$

Combining the two, we have in this case the Liapounoff–Haupt sequence

$$0 = \Omega_0 < \hat{\Omega}_1 = \hat{\Omega}_2 < \Omega_1 = \Omega_2 < \hat{\Omega}_3 = \hat{\Omega}_4 < \Omega_3 = \Omega_4 < \cdots, \tag{71}$$

for $\Gamma = 1$.

We now consider the case when $\Gamma \neq 1$, $\sigma = 1$. To find the zeros of multiplicity 2, we consider the product of eqns (69), which can be written in the form

$$\left(\sqrt{\Gamma} - \frac{1}{\sqrt{\Gamma}}\right)^2 \sin 2x \sin 2Tx - 2[\cos 2(1+T)x - 1] = 0. \tag{72}$$

The first derivative of this equation is

$$\left(\sqrt{\Gamma} - \frac{1}{\sqrt{\Gamma}}\right)^2 [\cos 2x \sin 2Tx + T \sin 2x \cos 2Tx] + 2(1+T) \sin 2(1+T)x = 0. \tag{73}$$

Evidently, these two equations will be satisfied if x and T are so chosen that

$$\sin 2x = 0, \sin 2Tx = 0, \text{and } \cos 2(1+T)x = 1. \tag{74}$$

The roots x_n so determined, are roots of multiplicity 2 and are given by

$$x_n = q\pi/2,\ \ T = p/q$$
$$(q+p) = 2n,\ \ q, p, n = 0, 1, 2, 3, \ldots . \tag{75}$$

We thus see that there exist multiple roots only when T is a rational number. For rational values of T, a line with this slope will pass through the intersection points of the spectral lines corresponding to the two equations (69). The coordinates of the intersection points are $(q(\pi/2), p(\pi/2))$, where at each intersection point $(q+p) = 2n$. These intersection points are clearly shown in Fig. 3, and all those intersection points which lie on the line passing through the origin with slope $T = p/q$, are zeros of multiplicity 2. Thus if $(q\pi/2, p\pi/2)$ where $(q+p) = 2n$ be the coordinates of the first zero of multiplicity 2 on the line $T = p/q$, then the coordinates of higher zeros on this line are $m(q\pi/2, p\pi/2)$, where $m(q+p) = 2n$. As a typical case, the Ω-sub-sequence for $T = 3/1$ is given by

$$0 = \Omega_0 < \Omega_1 < \Omega_2 < \Omega_3 = \Omega_4 < \Omega_5 < \Omega_6 < \Omega_7 = \Omega_8 < \Omega_9 < \cdots . \tag{76}$$

We now consider the second case when $\Gamma \neq 1$ and $\sigma = -1$. It can similarly be shown that corresponding to eqn (70), there exist double roots if x and T are so chosen that they satisfy the equations

$$\sin 2x = 0, \sin 2Tx = 0, \text{ and } \cos 2(1+T)x = -1. \tag{77}$$

In this case, we have roots of multiplicity 2 when

$$x_n = q\pi/2,\ \ T = p/q$$
$$(q+p) = 2n+1,\ \ q, p, n = 0, 1, 2, 3, \ldots \tag{78}$$

The coordinates of the intersection point of the family of curves corresponding to eqn (70) are $(q\pi/2, p\pi/2)$ where $(q+p) = 2n+1$. Those intersection points which lie on the line with slope $T = p/q$, have the coordinates $m(q\pi/2, p\pi/2)$, where $m(q+p) = 2n+1$. As a typical case, the Ω-sub-sequence for $T = 2/1$ is

$$\hat{\Omega}_1 < \hat{\Omega}_2 < \hat{\Omega}_3 = \hat{\Omega}_4 < \hat{\Omega}_5 < \hat{\Omega}_6 < \hat{\Omega}_7 < \hat{\Omega}_8 < \hat{\Omega}_9 = \hat{\Omega}_{10} < \cdots . \tag{79}$$

Combining the two results, the Liapounoff–Haupt sequence for a typical value of $T = p/q$ can now be easily constructed. Choosing $T = 3/1$, the sequence is

$$\begin{aligned} 0 = \Omega_0 < \hat{\Omega}_1 < \hat{\Omega}_2 < \Omega_1 < \Omega_2 < \hat{\Omega}_3 < \hat{\Omega}_4 < \Omega_3 = \Omega_4 \\ < \hat{\Omega}_5 < \hat{\Omega}_6 < \Omega_5 < \Omega_6 < \hat{\Omega}_7 < \hat{\Omega}_8 < \Omega_7 = \Omega_8 < \cdots . \end{aligned} \tag{80}$$

Now consider the intersection of the line with slope $T = p/q$, with the spectral lines corresponding to $\sigma = \pm 1$. At each intersection point there exists at least one periodic solution, of period d or $2d$, depending upon $\sigma = +1$, or -1, respectively. When the line with slope $T = p/q$ intersects a lattice-point with coordinates $(q\pi/2, p\pi/2)$, two periodic solutions, both of period d or of period $2d$, co-exist. This is called co-existence, implying existence of 2 periodic solutions of the same period, or coincidence, implying existence of eigenvalues of double multiplicity. Naturally, coincidence of frequencies implies co-existence of periodic solutions, and conversely.

We now consider the problem of co-existence of periodic solutions when the radial eigenvalue $\kappa^2 > 0$. The theory tells us that when $\sigma = \pm 1$, there exists, in general, one periodic solution corresponding to every simple eigenvalue of the characteristic equations (54) and (55). The other linearly independent solution corresponding to the same eigenvalue is, in general, aperiodic. The problem of co-existence of periodic solutions requires the existence of non-simple eigenvalues of double multiplicity. We therefore look for double roots of the characteristic equation when $\kappa^2>0$. When $\sigma=1$, $\kappa^2>0$, the characteristic equation (54) has the equivalent form

$$\left(\sqrt{\Lambda}-\frac{1}{\sqrt{\Lambda}}\right)^2 \sin 2\sqrt{[x^2-(\kappa\tau)^2]}\sin 2T\sqrt{[x^2-(\kappa\tau/\gamma)^2]}$$
$$-2[\cos 2(\sqrt{[x^2-(\kappa\tau)^2]}+T\sqrt{[x^2-(\kappa\tau/\gamma)^2]})-1]=0, \tag{81}$$

where

$$\Lambda \equiv \Gamma\sqrt{[(x^2-(\kappa\tau/\gamma)^2)/(x^2-(\kappa\tau)^2)]},$$
$$\Gamma \equiv \gamma\bar{\mu}/\mu,\ T\equiv \gamma\bar{\tau}/\tau,\ x\equiv \tau\Omega. \tag{82}$$

We consider the case when $1/\gamma>1$. Then eqn (81) and its first derivative will simultaneously be satisfied if x and T are so chosen that they satisfy the equations

$$\sin 2\sqrt{[x^2-(\kappa\tau)^2]}=0,\ \sin 2T\sqrt{[x^2-(\kappa\tau/\gamma)^2]}=0,$$
$$\cos 2(\sqrt{[x^2-(\kappa\tau)^2]}+T\sqrt{[x^2-(\kappa\tau/\gamma)^2]})=1. \tag{83}$$

The roots of multiplicity 2 are therefore given by

$$x_q=\sqrt{[(q\pi/2)^2+(\kappa\tau)^2]},\ T=\frac{p}{\sqrt{\left[q^2-(1-\gamma^2)\left(\frac{2\kappa\tau}{\pi\gamma}\right)^2\right]}},$$
$$p+q=2n,\ q,p,n=0,1,2,3,\ldots \tag{84}$$

We thus see that there exist roots x_q of multiplicity 2 if, in general, T is an appropriately chosen irrational number satisfying condition $(84)_2$. If for a given κ, there exist an integer l and parameters γ and τ such that $(1/\gamma^2-1)(2\tau/\pi)^2=(q^2-l^2)/\kappa^2$, then T is a rational periodic decimal number. If we assume that T is a suitably chosen irrational number, then a parabolic curve $T\sqrt{[x^2-(\kappa\tau/\gamma)^2]}$ will pass through a lattice-point $(x_q, p\pi/2)$ of the intersecting family of spectral lines corresponding to the two characteristic equations $(67)_{1,2}$. This lattice-point $(x_q, p\pi/2)$, $p+q=2n$, is then a zero of multiplicity 2. Thus for $\gamma=1/4$, $\tau=9\pi/20$, $\kappa_1=1.18920$, $x_5=2.556\ 663\pi$ will be a zero of multiplicity 2 if we choose $\bar{\tau}$ in such a way that $T=p/2.795\ 981$, where $5+p=2n$. Similarly $x_6=3.047\ 355\pi$ will be a zero of multiplicity 2 for $T=p/4.337916$, where $6+p=2n$. As an example, let $q=6$ and $p=2$ so that $4.337\ 916T=2$. The parabola $T\sqrt{[x^2-(\kappa\tau/\gamma)^2]}$ now intersects the lattice-point (x_6, π), and examination of Fig. 4 reveals that we have a subsequence which can be arranged in an ascending order, except that one of the eigenvalues is of multiplicity 2. The structure of this sequence is obvious from Fig. 4.

We now consider the case when $\sigma=-1$, $\kappa^2>0$. It can similarly be shown that corresponding to the characteristic equation (55), there exist double roots if x and T are so chosen that they satisfy the equations

$$\sin 2\sqrt{[x^2-(\kappa\tau)^2]}=0,\ \sin 2T\sqrt{[x^2-(\kappa\tau/\gamma)^2]}=0,$$
$$\cos 2(\sqrt{[x^2-(\kappa\tau)^2]}+T\sqrt{[x^2-(\kappa\tau/\gamma)^2]})=-1. \tag{85}$$

In this case the roots of multiplicity 2 are given by

$$x_q=\sqrt{[(q\pi/2)^2+(\kappa\tau)^2]},\ T=\frac{p}{\sqrt{\left(q^2-(1-\gamma^2)\left(\frac{2\kappa\tau}{\pi\gamma}\right)^2\right)}}, \tag{86}$$
$$p+q=2n+1,\ p,q,n=0,1,2,3,\ldots$$

The discussion concerning these roots of multiplicity 2, follows along the same lines as in the previous case when $\sigma = +1$, and therefore hardly needs reiteration. However, we may repeat that existence of the double roots implies co-existence of two periodic solutions, both of the same period, d or $2d$, whichever the case may be.

9. STABILITY

To study regions of lability and stability we now plot graphs showing x vs T, for $\kappa_0 = 0$ and $\kappa_1 = 1.18920$, which are shown in Figs. 5 and 6, respectively.

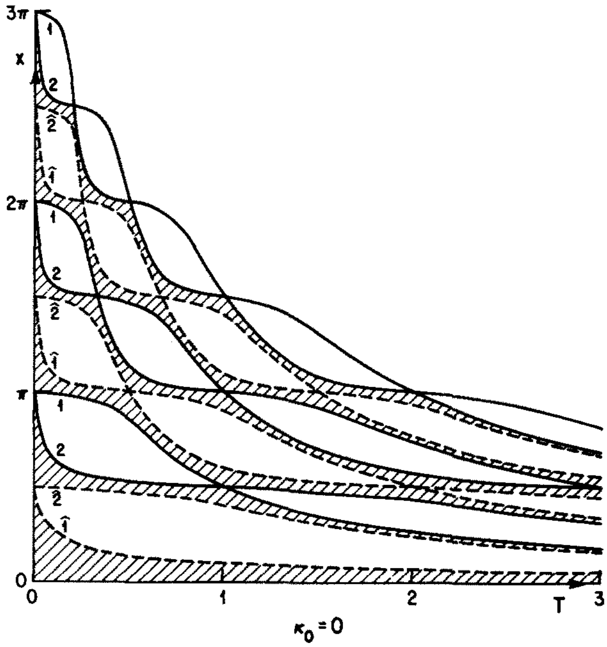

Fig. 5. Stability chart x vs T for pure torsional modes, $\kappa_0 = 0$. Regions of stability are shown hatched where the solutions are bounded. Periodic solutions co-exist at points of intersection.

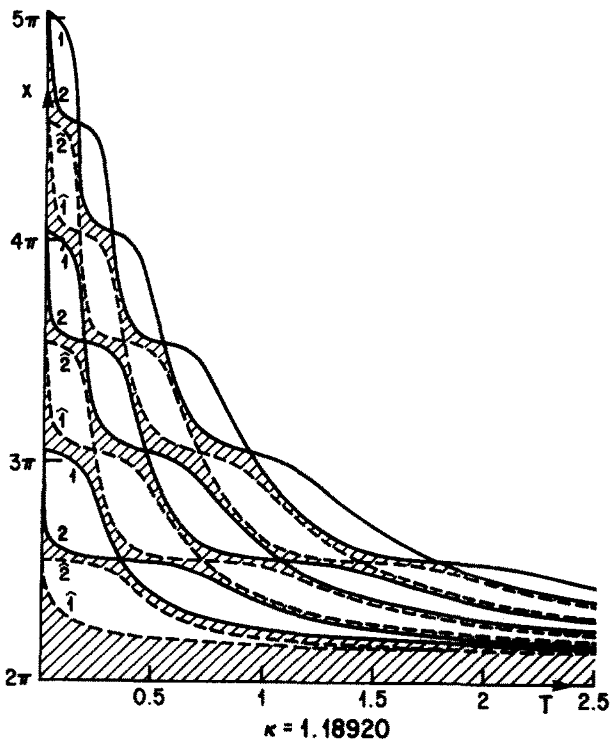

Fig. 6. Stability chart x vs T for torsional modes with one radial node, $\kappa_1 = 1.18920$. Regions of stability are shown hatched where the solutions are stable. Periodic solutions co-exist at points of intersection.

Solutions are called stable and bounded when $|H| < 2$. For values of x and T in the hatched regions in these figures, all solutions are stable, and thus the hatched regions are called regions of stability. Unhatched regions are regions of lability where $|H| > 2$, and the solutions are unbounded. On the boundaries of hatched and unhatched regions, $|H| = 2$, and therefore for a typical point (x, T) on the boundary, there exists at least one periodic solution of period d, or $2d$. At a point of intersection, two such periodic solutions co-exist. These points of intersection, correspond to similar intersection points in Figs. 3 and 4. We note from the stability diagrams that there is one value of T for which $\Omega_1 = \Omega_2$, three values of T for which $\Omega_3 = \Omega_4$, 5 values of T for which $\Omega_5 = \Omega_6$, etc..., no value of T for which $\hat{\Omega}_1 = \hat{\Omega}_2$, 2 values of T for which $\hat{\Omega}_3 = \hat{\Omega}_4$, 4 values of T for which $\hat{\Omega}_5 = \hat{\Omega}_6$, etc.

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